

# On an $AF$ -algebra of the Hecke eigenform

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## Abstract

An  $AF$ -algebra  $\mathbb{A}_f$  is assigned to each cusp form  $f$  of weight two; we study properties of this operator algebra, when  $f$  is the Hecke eigenform.

*Key words and phrases:* cusp forms,  $AF$ -algebras

*AMS (MOS) Subj. Class.:* 11F03; 46L85

## 1 Introduction

**A. The Hecke eigenforms.** Let  $N > 1$  be a natural number and consider a (finite index) subgroup of the modular group given by the formula:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Let  $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$  be the upper half-plane and let  $\Gamma_0(N)$  act on  $\mathbb{H}$  by the linear fractional transformations; consider an orbifold  $\mathbb{H}/\Gamma_0(N)$ . To compactify the orbifold at the cusps, one adds a boundary to  $\mathbb{H}$ , so that  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$  and the compact Riemann surface  $X_0(N) = \mathbb{H}^*/\Gamma_0(N)$  is called a *modular curve*. A collection of the meromorphic functions  $f(z)$  on  $\mathbb{H}$ , which vanish at the cusps and such that:

$$f\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2} f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

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\*Partially supported by NSERC.

is called *cuspidal forms* of weight two; the (complex linear) space of such forms will be denoted by  $S_2(\Gamma_0(N))$ . The formula  $f(z) \mapsto \omega = f(z)dz$  defines an isomorphism  $S_2(\Gamma_0(N)) \cong \Omega_{hol}(X_0(N))$ , where  $\Omega_{hol}(X_0(N))$  is the space of holomorphic differentials on the Riemann surface  $X_0(N)$ . Note that  $\dim_{\mathbb{C}}(S_2(\Gamma_0(N))) = \dim_{\mathbb{C}}(\Omega_{hol}(X_0(N))) = g$ , where  $g = g(N)$  is the genus of the surface  $X_0(N)$ . A Hecke operator,  $T_n$ , acts on  $S_2(\Gamma_0(N))$  by the formula  $T_n f = \sum_{m \in \mathbb{Z}} \gamma(m) q^m$ , where  $\gamma(m) = \sum_{a|GCD(m,n)} a c_{mn/a^2}$  and  $f(z) = \sum_{m \in \mathbb{Z}} c(m) q^m$  is the Fourier series of the cusp form  $f$  at  $q = e^{2\pi iz}$ . The  $T_n$  is a self-adjoint linear operator on the vector space  $S_2(\Gamma_0(N))$  endowed with the Petersson inner product; the algebra  $\mathbb{T}_N := \mathbb{Z}[T_1, T_2, \dots]$  is a commutative algebra. Any cusp form  $f_N \in S_2(\Gamma_0(N))$ , which is an eigenvector for one (and hence all) of  $T_n$ , is referred to as a *Hecke eigenform*; such an eigenform is called *rational*, whenever its Fourier coefficients  $c(m) \in \mathbb{Z}$ .

**B. An AF-algebra  $\mathbb{A}_f$ .** Let  $f \in S_2(\Gamma_0(N))$  be a cusp form and  $\omega = f dz$  the corresponding holomorphic differential on  $X_0(N)$ . We shall denote by  $\phi = \text{Re } \omega$  a closed 1-form on  $X_0(N)$  and consider its periods  $\lambda_i = \int_{\gamma_i} \phi$  against a basis  $\gamma_i$  in the (relative) homology group  $H_1(X_0(N), Z(\phi); \mathbb{Z})$ , where  $Z(\phi)$  is the set of zeros of  $\phi$ . Assume  $\lambda_1 \neq 0$  and take a vector  $\theta = (\theta_1, \dots, \theta_{n-1})$  with  $\theta_i = \lambda_{i+1}/\lambda_1$ . The Jacobi-Perron continued fraction of  $\theta$  ([2]) is given by the formula:

$$\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{i \rightarrow \infty} \begin{pmatrix} 0 & 1 \\ I & b_i \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_i \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} = \lim_{i \rightarrow \infty} B_i \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix},$$

where  $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$  is a vector of the non-negative integers,  $I$  the unit matrix and  $\mathbb{I} = (0, \dots, 0, 1)^T$ . By  $\mathbb{A}_f$  we shall understand an (isomorphism class of) *AF-algebra* given the Bratteli diagram with the partial multiplicity matrices  $B_i$ . Recall, that the *AF-algebra* is called *stationary*, if  $B_i = B = \text{Const}$  [5].

**C. The result.** Let  $\mathbb{A}_{f_N}$  be an *AF-algebra*, such that  $f_N \in S_2(\Gamma_0(N))$  is a Hecke eigenform. Our main result can be stated as follows.

**Theorem 1** *The  $\mathbb{A}_{f_N}$  is a stationary AF-algebra, unless  $f_N$  is a rational eigenform, in which case  $\mathbb{A}_{f_N} \cong \mathbb{C}$ .*

Theorem 1 is proved in section 2. Since our note does not include a formal section on the preliminaries, we encourage the reader to consult [5] (*AF-algebras*), [4] (*cuspidal forms*) and [2] (*the Jacobi-Perron continued fractions*).

## 2 Proof

A standard dictionary ([5]) between the  $AF$ -algebras and their dimension groups is adopted. Instead of dealing with  $\mathbb{A}_f$ , we work with its dimension group  $G_{\mathbb{A}_f} = (G, G^+)$ , where  $G \cong \mathbb{Z}^n$  is the lattice and  $G^+ = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid \theta_1 x_1 + \dots + \theta_{n-1} x_{n-1} + x_n \geq 0\}$  is a positive cone. Recall, that  $G_{\mathbb{A}_f}$  is abelian group with an order, which defines the  $AF$ -algebra  $\mathbb{A}_f$ , up to a stable isomorphism. We arrange the proof in a series of lemmas. First, let us show, that  $\mathbb{A}_f$  is a correctly defined  $AF$ -algebra.

**Lemma 1** *The  $\mathbb{A}_f$  does not depend, up to a stable isomorphism, on a basis in  $H_1(X_0(N), Z(\phi); \mathbb{Z})$ .*

*Proof.* Denote by  $\mathfrak{m} := \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n$  a  $\mathbb{Z}$ -module in the real line  $\mathbb{R}$ . Let  $\{\gamma'_i\}$  be a new basis in  $H_1(X_0(N), Z(\phi); \mathbb{Z})$ , such that  $\gamma'_i = \sum_{j=1}^n a_{ij} \gamma_j$  for a matrix  $A = (a_{ij}) \in GL_n(\mathbb{Z})$ . Using the integration rules, one gets:  $\lambda'_i = \int_{\gamma'_i} \phi = \int_{\sum_{j=1}^n a_{ij} \gamma_j} \phi = \sum_{j=1}^n \int_{\gamma_j} \phi = \sum_{j=1}^n a_{ij} \lambda_j$ . Thus,  $\mathfrak{m}' = \mathfrak{m}$  and the change of basis in the homology group  $H_1(X_0(N), Z(\phi); \mathbb{Z})$  amounts to a change of basis in the module  $\mathfrak{m}$ . It is an easy exercise to show, that there exists a linear transformation of  $\mathbb{Z}^n$  sending the positive cone  $G^+$  of  $G_{\mathbb{A}_f}$  to the positive cone  $(G^+)'$  of  $G_{\mathbb{A}'_f}$ . In other words,  $\mathbb{A}'_f$  and  $\mathbb{A}_f$  are stably isomorphic.  $\square$

Let  $f_N \in S_2(\Gamma_0(N))$  be the Hecke eigenform; the Fourier coefficients  $c(m)$  of  $f_N$  are algebraic integers and we denote by  $\mathbb{K}_{f_N} = \mathbb{Q}(c(m))$  an extension of the field  $\mathbb{Q}$  by the Fourier coefficients of  $f_N$ . The  $\mathbb{K}_{f_N}$  is a real algebraic number field of degree  $1 \leq \deg(\mathbb{K}_{f_N}|\mathbb{Q}) \leq g$ , where  $g$  is the genus of surface  $X_0(N)$ . Along with  $\mathbb{K}_{f_N}$ , consider an  $n$ -dimensional abelian variety  $A_{f_N} = \mathbb{C}^n / (\mathbb{Z}z_1 + \dots + \mathbb{Z}z_n)$ , where  $z_i = \int_{\gamma_i} \omega_N$  are the (complex) periods of  $\omega_N$  against a basis  $\gamma_i$  in  $H_1(X_0(N); \mathbb{Z})$ . It is well known, that  $n = \deg(\mathbb{K}_{f_N}|\mathbb{Q})$ , cf. [4], Proposition 6.6.4.

**Lemma 2** *The (scaled) periods  $\lambda_i$  belong to the field  $\mathbb{K}_{f_N}$ .*

*Proof.* Let  $\mathfrak{m} = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_{2g}$  be a  $\mathbb{Z}$ -module generated by  $\lambda_i$ ; we seek the effect of the Hecke operators  $T_m$  on  $\mathfrak{m}$ . By the definition of a Hecke eigenform,  $T_m f_N = c(m) f_N$  for all  $T_m \in \mathbb{T}_N$ . In view of the isomorphism  $S_2(\Gamma_0(N)) \cong \Omega_{hol}(X_0(N))$ , one gets  $T_m \omega_N = c(m) \omega_N$ , where  $\omega_N = f_N dz$ . Then  $Re(T_m \omega_N) = T_m(Re \omega_N) = Re(c(m) \omega_N) = c(m) Re \omega_N$ . Therefore,  $T_m \phi_N = c(m) \phi_N$ , where  $\phi_N = Re \omega_N$ . The action of  $T_m$  on  $\mathbb{Z}$ -module

$\mathfrak{m}$  can be written as  $T_m(\mathfrak{m}) = \int_{H_1} T_m \phi_N = \int_{H_1} c(m) \phi_N = c(m) \mathfrak{m}$ , where  $H_1 := H_1(X_0(N), Z(\phi_N); \mathbb{Z})$ . Thus, the Hecke operator act on the module  $\mathfrak{m}$  as multiplication by an algebraic integer  $c(m) \in \mathbb{K}_{f_N}$ .

The action of  $T_m$  on  $\mathfrak{m} = \mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n$  can be written as  $T_m \lambda = c(m) \lambda$ , where  $\lambda = (\lambda_1, \dots, \lambda_n)$ ; thus,  $T_m$  is a linear operator (on the space  $\mathbb{R}^n$ ), whose eigenvector  $\lambda$  corresponds to the eigenvalue  $c(m)$ . It is an easy exercise in linear algebra, that  $\lambda$  can be scaled so that  $\lambda_i$  lie in the same field as  $c(m)$ ; lemma 2 follows.  $\square$

**Case I.** Let  $f_N$  be an irrational eigenform; then  $n = \deg(\mathbb{K}_{f_N}|\mathbb{Q}) \geq 2$ . Note, that  $\mathbb{Z}\lambda_1 + \dots + \mathbb{Z}\lambda_n$  is a full (i.e. maximal rank)  $\mathbb{Z}$ -module in the number field  $\mathbb{K}_{f_N}$ . Indeed,  $\text{rank}(\mathfrak{m}) \leq n$ , since  $z_{n+1} = \sum_{i=1}^n a_i z_i$  ( $a_i \in \mathbb{Z}$ ) and  $\lambda_i = \text{Re } z_i$ . On the other hand,  $A_{f_N}$  is an irreducible variety, in the sense that there are no Hecke eigenforms  $f'_N \in S_2(\Gamma_0(N))$ , such that  $\mathbb{K}_{f'_N}$  is a subfield of  $\mathbb{K}_{f_N}$ ; thus,  $\text{rank}(\mathfrak{m}) = n$ .

**Lemma 3** *The vector  $(\lambda_1, \dots, \lambda_n)$  has a periodic (Jacobi-Perron) continued fraction.*

*Proof.* Since  $\mathfrak{m} \subset \mathbb{K}_{f_N}$  is a full  $\mathbb{Z}$ -module, its endomorphism ring  $\text{End}(\mathfrak{m}) = \{\alpha \in \mathbb{K}_{f_N} : \alpha \mathfrak{m} \subseteq \mathfrak{m}\}$  is an order (a subring of the ring of integers) of the number field  $\mathbb{K}_{f_N}$ ; let  $u$  be a unit of the order [3], p 112. The action of  $u$  on  $\mathfrak{m}$  can be written in a matrix form  $A\lambda = u\lambda$ , where  $\lambda$  is a basis in  $\mathfrak{m}$  and  $A \in GL_n(\mathbb{Z})$ ; with no loss of generality, one can assume matrix  $A$  to be non-negative in a proper basis of  $\mathfrak{m}$ .

According to [1], Prop.3, the matrix  $A$  can be uniquely factorized as  $A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}$ , where vectors  $b_i = (b_1^{(i)}, \dots, b_{n-1}^{(i)})^T$  have non-negative integer entries. By [6], Satz XII, the periodic continued fraction

$$\begin{pmatrix} 1 \\ \theta' \end{pmatrix} = \text{Per} \overline{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix}} \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix} \quad (1)$$

converges to a vector  $\lambda' = (\lambda'_1, \dots, \lambda'_n)$ , which satisfies the equation  $A\lambda' = u\lambda'$ . Since  $A\lambda = u\lambda$ , the vectors  $\lambda$  and  $\lambda'$  are collinear; but collinear vectors have the same continued fractions [2].  $\square$

The first case of theorem 1 follows from lemma 3, since  $A_{f_N}$  is a stationary  $AF$ -algebra, whose period is given by the matrix  $A$ .

Case II. Let  $f_N$  be a rational eigenform; in this case  $n = 1$  and  $\mathbb{K}_{f_N} = \mathbb{Q}$ . The Bratteli diagram of  $\mathbb{A}_{f_N}$  is finite and one-dimensional; therefore,  $\mathbb{A}_{f_N} \cong M_1(\mathbb{C}) = \mathbb{C}$ . This argument finishes the proof of theorem 1.  $\square$

### 3 Final remarks

The  $\mathbb{A}_{f_N}$  is an interesting  $AF$ -algebra on its own right; aside, it captures geometry of the abelian variety  $A_{f_N}$ , which is critical for modularity theory of the rational elliptic curves [4]. This paper is preparatory for a forthcoming study of  $K$ -theory of the crossed product of  $\mathbb{A}_{f_N}$  by the endomorphisms, generated by the Hecke operators  $T_n$ .

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